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21. Computation of 4-layer apparent resistivity curves using a small computer.

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INTRODUCTION

A graphical approach has long been used in the interpretation of apparent resistivity curves and several sets of curves for two to four layers are available (*e.g.* Compagnie Générale de Géophysique, 1955; Mooney & Wetzel, 1956; Flathe, 1963). Theoretical curves for models of more than three layers are approximated using published nomograms. Although such 'type' curves have considerable use in making qualitative interpretations they rarely permit detailed interpretation and are inadequate to deal with problems involving thin layers of high contrast.

The advent of the high speed computer has changed this situation to the extent that it is now possible to calculate the curve for a given model (*e.g.* Cook, 1969). However cost is an important factor when considering the availability or use of such a machine. This factor, which has been discussed by Cook (1969), is due principally to the difficulties of generating a series expansion in two parts and then comparing coefficients throughout the storage. Much time is expended in constructing the parameters of the image points for the depths and resistivities desired. As a result the method and programme of Cook (1969) are of limited use where funds are restricted, expecially if a calculation to a high order approximation is desired. Further, storage of the large number of terms generated is restricted even in the largest machines.

Quantitative resistivity interpretations have been held in abeyance by the author for many years after it became apparent that the published curves were quite inadequate to deal with Tasmanian conditions. Interpretations have thus been general, incomplete and geologically sketchy. It is usually uneconomic to use general programmes such as that of Cook (1969). Upon receipt of a small high speed desk computer (Wang 700B) with an accurate plotter facility it was decided to re-examine the basic theory in the hope of finding an accurate solution to the problem in a form that could be handled by a smaller machine. It was obvious from the start that some limitations would be imposed although storage would probably not be one of them (compare Cook, 1969) due to the flexibility of programme arrangement to fully utilise a smaller storage.

An examination of various approaches to the computation showed that the method of Van Dam (1965) which was designed for hand calculation would be the most suitable. However in converting the method for automatic use it was desired to improve accuracy, calculate the complete series for each approximation and build in any correction for series convergence deficiencies.

The primary considerations at this point were to decide:

(1) The number of layers to be considered; and

(2) the degree of approximation required.

Experience has shown that most curves obtained in Tasmania are three or four layer types. Very rarely is there anything more complex. There was thus no need to provide for solutions to conditions exceeding four layers. Thus the more general approach of Cook (1969) is unnecessary; it is also beyond the capability of desk computers. Choosing to provide for a four layer condition means that 2 or 3 layer conditions are automatically covered and that a 5 or 6 layer situation could be built up with reasonable accuracy by first calculating a model for the first four layers and then for the second four and matching the overlapping region. This process is employed in the use of standard graphs but is there restricted by the specialisation inherent in the parameters upon which the curves are produced.

Examination of the examples and theory given by Van Dam (1965) shows that the third degree of approximation will be more than adequate for a four layer case if allowance is made for the degree of series convergence and appropriately compensated for in calculation.

The programme has been written to provide for the Schlumberger configuration. Variation of any proposition used is discussed in the conclusion.

MATHEMATICAL METHOD

The following is a summary of the theory presented by Van Dam (1965). The surface potential distribution due to a pole at the surface of resistivities ρ_m and lower boundary depths d_m was given by Stefanescu. The potential V at a point distant r from a pole emitting current I is given by:

$$V = \rho_1 \frac{I}{2\pi}\left\{\frac{1}{r} + 2\int_0^\infty \theta_1(\lambda) J_0(\lambda r)\, d\lambda\right\}$$

where λ = an integration variable,

J_0 = Bessel function of first kind and zero order

θ_1 = kernel function determined by boundary conditions. Subscript refers to first layer.

The kernel function $\theta_n(\lambda)$ is a function of λ and the reflection coefficients.

$$k_m = \frac{\rho_{m+1} - \rho_m}{\rho_{m+1} + \rho_m}$$

at the interfaces and of the depths d_m of the interfaces $(m<n)$

$$\theta_n(\lambda) = \frac{P_n(u)}{Q_n(u)} = \frac{P_n(u)}{H_n(u) - P_n(u)} \quad \text{where } u = e^{-2\lambda}$$

$\theta_n(\lambda)$ can be expanded as an infinite series such that

$$\frac{P_n(u)}{Q_n(u)} = P_n(u)\left\{1 + [1-Q_n(u)] + [1-Q_n(u)]^2 + [1-Q_n(u)]^3 \ldots\ldots\right\}$$

which converges provided $0<Q_n(u)<2$

For two layers: $P_2(u) = k_1 u^{d_1}$, $H_2(u) = 1$, $Q_2(u) = 1 - k_1 u^{d_1}$ and by 'law of formation', Flathe (1955).

$$P_n(u) = P_{n-1}(u) + H_{n-1}(u^{-1}) k_{n-1} u^{d_{n-1}}$$

$$H_n(u) = H_{n-1}(u) + P_{n-1}(u^{-1}) k_{n-1} u^{d_{n-1}}$$

$$Q_n(u) = Q_{n-1}(u) - Q_{n-1}(u^{-1}) k_{n-1} u^{d_{n-1}}$$

Thus in the p^{th} degree of approximation to the series the expression for V according to Stefanescu can be replaced by

$$V = \rho_1 \frac{I}{2\pi}\Big[\frac{1}{r} + 2\int_0^\infty P_n(u) J_o(\lambda r)\, d\lambda + 2\int_0^\infty P_n(u)\Big\{1-Q_n(u)\Big\} J_o(\lambda r)\, d\lambda$$
$$+2\int_0^\infty P_n(u)\Big\{1-Q_n(u)\Big\}^2 J_o(\lambda r)\, d\lambda \ldots\ldots$$
$$+2\int_0^\infty P_n(u)\Big\{1-Q_n(u)\Big\}^{p-1} J_o(\lambda r)\, d\lambda\Big]$$

$P_n(u)$, $\left\{1-Q_n(u)\right\}$ are both polynomials in $u = e^{-2\lambda}$. Thus any product of the form $P_n(u)\left\{1-Q_n(u)\right\}$ is built up of a finite number of terms of the form:

$Ke^{-D\lambda}$ where K is a function of the reflection coefficients k_m and D is a linear function of depths d_m. Each K and D define an image pole of appropriate strength and depth. For example the term $k_1 u^{d1} = k_1 e^{-2\lambda d1}$, yields the image pole $K = 2k_1$, $D = 2d_1$.

Hence $V = \rho_1 \frac{I}{2\pi}\left\{\frac{1}{r} + \text{an infinite number of terms } \frac{K}{\sqrt{D^2+r^2}}\right\}$

In any successive degree of approximation, p, another group of terms is added. For any given n, the number of terms increases with p.

Relating this to the Schlumberger configuration:

$$\rho a = \frac{-2\pi}{I} r^2 \frac{dv}{dr}$$
$$= \rho_1\left\{1+\sum K \frac{1}{\sqrt{(D/r)^2+1}}^{\,3}\right\}$$

COMPUTATION METHOD

The procedure used for calculation requires development of the particular series based on $n = 4$ and $p = 3$. Provided $0<Q_n(u)<2$ convergency is assured although the percentage error at $\lambda = 1$ may still be large. If $Q_n(u)$ is only slowly convergent or even divergent convergence may be forced in order to provide for high accuracy. Greatest accuracy is achieved where $Q_n(u)$ is close to 1.

Van Dam (1965) shows that it is possible to select a factor a such that $a.Q_n(u)$ as a whole is as close to 1 as possible. Such a factor in no way affects the kernel function but does apply variation to the terms of the series. For terms derived from the first degree of approximation this process leads to image strengths being multiplied by a; in the second degree by $(2a-a^2)$ for terms due to first degree and a^2 for those of second degree only and for third degree series by $(3a-3a^2+a^3)$ for terms based on $p = 1$, $(3a^2-2a^3)$ for terms based on $p = 2$, by a a^3 for those added for $p = 3$. These factors apply to the overall third degree approximation.

For $n = 4$, $p = 3$.

$$P_4(u) = k_1 u^{d_1} + k_2 u^{d_2} + k_3 u^{d_3} + k_1 k_2 k_3 u^{d_3-d_2+d_1}$$

$$Q_4(u) = 1 - k_1 u^{d_1} - k_2 u^{d_2} - k_3 u^{d_3} + k_1 k_2 u^{d_2-d_1} + k_1 k_3 u^{d_3-d_1}$$
$$+ k_2 k_3 u^{d_3-d_2} - k_1 k_2 k_3 u^{d_1-d_2+d_3}$$

Test: $O<Q_4(u)<2$ for $\lambda = 0, 0.1, 0.2$ ...... 1.

Intervals of 0.1λ have been chosen as being sufficiently representative to test $Q_4(u)<2$: it being unlikely that $Q_n(u)>2$ would occur without indication between values (see discussion). Examination of the degree of convergence across the range yields the appropriate value of the factor to be applied to give the highest accuracy.

The terms of the series are derived from:

$$2\int_0^\infty P_4(u)J_o(\lambda r)\,d\lambda+2\int_0^\infty P_4(u)\left\{1-Q_4(u)\right\}J_o(\lambda r)\,d\lambda+2\int_0^\infty P_4(u)\left\{1-Q_4(u)\right\}^2 J_o(\lambda r)\,d\lambda$$

$n = 4$

Substitution and expansion yields poles:

	$p = 1$	$p = 2$
$2d_1$	$2k_1$	
$4d_1$		$2k_1{}^2$
$2d_2$	$2k^2$	$-2k_1{}^2k_2$
$2d_2+2d_1$		$4k_1k_2$
$4d_2$		$2k_2{}^2$
$4d_2-2d_1$		$-2k_1k_2{}^2$
$2d_3+2d_2$		$4k_2k_3$
$2d_3$	$2k_3$	$-2k_1{}^2k_3, -2k_1{}^2k_2{}^2k_3, -2k_2{}^2k_3$
$2d_3+2d_1$		$4k_1k_3, 4k_1k_2{}^2k_3$
$2d_3-2d_2+4d_1$		$4k_1{}^2k_2k_3$
$2d_3-2d_2+2d_1$	$2k_1k_2k_3$	$-2k_1k_2k_3$
$2d_3+2d_2-2d_1$		$-4k_1k_2k_3$
$4d_3-2d_1$		$-2k_1k_3{}^2$
$4d_3$		$2k_3{}^2$
$4d_3-2d_2$		$-2k_2k_3{}^2, -2k_1{}^2k_2k_3{}^2$
$4d_3-2d_2+2d_1$		$4k_1k_2k_3{}^2$
$4d_3-4d_2+2d_1$		$-2k_1k_2{}^2k_3{}^2$
$4d_3-4d_2+4d_1$		$2k_1{}^2k_2{}^2k_3{}^2$

$p = 3$

$6d_1$	$2k_1{}^3$
$2d_2+4d_1$	$6k_1{}^2k_2$
$2d_2+2d_1$	$-4k_1{}^3k_2$
$4d_2$	$-8k_1{}^2k_2{}^2$
$4d_2+2d_1$	$6k_1k_2{}^2$
$4d_2-2d_1$	$2k_1{}^3k_2{}^2$
$6d_2-2d_1$	$-4k_1k_2{}^3$
$6d_2-4d_1$	$2k_1{}^2k_2{}^3$
$6d_2$	$2k_2{}^3$

p = 3 - *continued*

$2d_3+4d_2$	$6k_2^2k_3$			
$2d_3+2d_2$	$-8k_1^2k_2^3k_3$	$-4k_2^3k_3$	$-16k_1^2k_2k_3$	
$2d_3$	$4k_1^2k_2^2k_3$			
$2d_3+2d_1$	$-8k_1^3k_2^2k_3$	$-4k_1^3k_3$	$-8k_1k_2^2k_3$	
$2d_3+4d_1$	$12k_1^2k_2^2k_3$	$6k_1^2k_3$		
$2d_3-2d_2+4d_1$	$-4k_1^2k_2k_3$			
$2d_3-2d_2+6d_1$	$6k_1^3k_2k_3$			
$2d_3+2d_2+2d_1$	$12k_1k_2k_3$	$6k_1k_2^3k_3$		
$2d_3+2d_2-2d_1$	$2k_1^3k_2^3k_3$	$4k_1k_2^3k_3$	$4k_1^3k_2k_3$	
$2d_3+4d_2-2d_1$	$-12k_1k_2^2k_3$			
$2d_3+4d_2-4d_1$	$6k_1^2k_2^2k_3$			
$4d_3+2d_2-4d_1$	$6k_1^2k_2k_3^2$			
$4d_3+2d_2-2d_1$	$-12k_1k_2k_3^2$			
$4d_3+2d_2$	$6k_2k_3^2$			
$4d_3$	$-8k_2^2k_3^2$	$-8k_1^2k_3^2$	$-16k_1^2k_2^2k_3^2$	
$4d_3+2d_1$	$6k_1k_3^2$	$12k_1k_2^2k_3^2$		
$4d_3-2d_2+2d_1$	$-8k_1k_2k_3^2$	$-8k_1^3k_2k_3^2$	$-8k_1k_2^3k_3^2$	$-4k_1^3k_2^3k_3^2$
$4d_3-2d_2$	$2k_2^3k_3^2$	$4k_1^2k_2k_3^2$	$4k_1^2k_2^3k_3^2$	
$6d_3-2d_2$	$-4k_2k_3^3$	$-8k_1^2k_2k_3^2$		
$6d_3-2d_2-2d_1$	$2k_1^3k_2k_3^3$	$4k_1k_2k_3^3$		
$6d_3-2d_1$	$-4k_1k_3^3$			
$4d_3-2d_1$	$2k_1^3k_3^2$	$8k_1k_2^2k_3^2$	$4k_1^3k_2^2k_3^2$	
$4d_3-2d_2+4d_1$	$12k_1^2k_2k_3^2$	$6k_1^2k_2^3k_3^2$		
$4d_3-4d_2+2d_1$	$2k_1k_2^2k_3^2$			
$4d_3-4d_2+4d_1$	$-8k_1^2k_2^2k_3^2$			
$4d_3-4d_2+6d_1$	$6k_1^3k_2^2k_3^2$			
$6d_3-6d_2+6d_1$	$2k_1^3k_2^3k_3^3$			
$6d_3-6d_2+4d_1$	$-4k_1^2k_2^3k_3^3$			
$6d_3-4d_2+4d_1$	$6k_1^2k_2^2k_3^3$			
$6d_3-4d_2+2d_1$	$-8k_1k_2^2k_3^3$	$-4k_1^3k_2^2k_3^3$		
$6d_3-6d_2+2d_1$	$2k_1k_2^3k_3^3$			
$6d_3-4d_2$	$2k_2^2k_3^3$	$4k_1^2k_2^2k_3^3$		
$6d_3-2d_2+2d_1$	$6k_1k_2k_3^3$			
$6d_3$	$2k_3^3$			
$6d_3-4d_1$	$2k_1^2k_3^3$			

PROGRAMME

The programme is not presented in detail as it is written for the Wang 700B in machine code. Copies of the programme are available in printed form

rho 1	rho 2	rho 3	rho 4	k 1	k 2	k 3
1000.00	250.00	62.50	187.50	-.60	-.60	.50
11.00	55.00	1.50	2.60	.66	-.94	.26
100.00	2000.00	50.00	1000.00	.90	-.95	.90

RESISTIVITY:

(Four layer Schlumberger)
Rho 100.00 2000.00 50.00 1000.00
Depth 1.0 40.0 240.0
Convergence; 1 tenth increment
.0176 .2592 .3935 .5034 .5934 -.1266 -.7575 -.4960 -.2964 -.1396 -.0134 A
Factor = 1.600 Capacity of machine exceeded in convergence routine - see text

RESISTIVITY:

(Four layer Schlumberger)
Rho 1000.00 250.00 62.50 187.50
Depth 1.0 4.0 6.0
Convergence; 1 tenth increment
1.2800 1.3975 1.3567 1.2945 1.2402 1.1968 1.1622 1.1342 1.1112 1.0921 1.0763 B
Factor = .781

RESISTIVITY

(Four layer Schlumberger)
Rho 11.00 55.00 1.50 2.60
Depth 5.0 72.0 154.0 C
Convergence; 1 tenth increment
.4748 .7547 .9097 .9668 .9877 .9955 .9983 .9993 .2201 .3069 .3762
Factor = 1.200

Figure 33. *Typical side calculation print-out for k values and heading blocks for each curve with convergence test.*

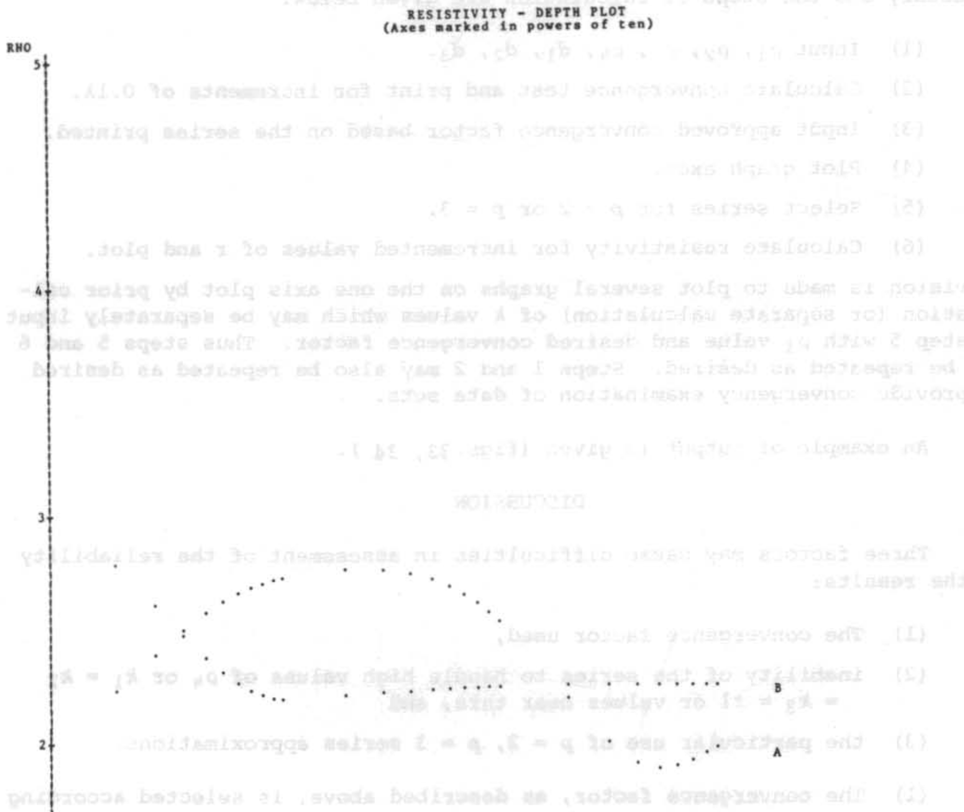

Figure 34. *Plot of some resistivity curves.*

5 cm

on request. As the programme flow is direct and simple a flow chart is unnecessary and the steps of calculation are given below:

(1) Input ρ_1, ρ_2, ρ_3, ρ_4, d_1, d_2, d_3.

(2) Calculate convergence test and print for increments of 0.1λ.

(3) Input approved convergence factor based on the series printed.

(4) Plot graph axes.

(5) Select series for $p = 2$ or $p = 3$.

(6) Calculate resistivity for incremented values of r and plot.

Provision is made to plot several graphs on the one axis plot by prior calculation (or separate calculation) of k values which may be separately input at step 5 with ρ_1 value and desired convergence factor. Thus steps 5 and 6 may be repeated as desired. Steps 1 and 2 may also be repeated as desired to provide convergency examination of data sets.

An example of output is given (figs.33, 34).

DISCUSSION

Three factors may cause difficulties in assessment of the reliability of the results:

(1) The convergence factor used,

(2) inability of the series to handle high values of ρ_4 or $k_1 = k_2 = k_3 = \pm 1$ or values near this, and

(3) the particular use of $p = 2$, $p = 3$ series approximations.

(1) The convergence factor, as described above, is selected according to the nature of the series convergence. In most cases examined, even with very wide variations in ρ and d values there is usually a strong convergence toward 1. However in extreme cases (*e.g.* $\rho_1 = 100$, $\rho_2 = 0.5$, $\rho_3 = 1{,}000$, $\rho_4 = 90{,}000$, $d_1 = 1$, $d_2 = 70$, $d_3 = 127$) the convergence may be as follows: 0.0, 1.81, 1.66, 1.54, 1.44, 1.36, 1.29, 1.24, 1.19, 1.15. A reasonable convergence factor might be about 0.7. However as shown in Figure 35 the series breaks down at $r = 100$. Variation of the factor between 0.6 and 1 does not solve the problem as indicated in the figure. Re-examination of the convergence test indicates discontinuity and failure for $0<\lambda<0.1$ and that $Q_n(u)$ does in fact exceed 2. Discontinuous or excessive convergence may be indicated by a convergence test that displays changes from either side of 1. A normal convergence will either be up or down to 1 with no excessive jumps either side of 1. The above example shows an extreme jump.

In the case of normal convergence a variation of ± 0.1 in factor produces no significant variation in result (fig. 37). Wide variation in factor value causes series failure (fig. 37), but any reasonably judged factor introduces very little error. Choice of the convergence may be critical where k_1 is large and negative (see fig. 38) and especially so when k_2 is also negative. Experience suggests however that the critical region is always $0.05<\lambda<0.25$ and values in this range, when multiplied by the factor, should be about 1. If the correct factor is chosen it has been found that the results obtained, (even for as low a degree of approximation as $p = 2$, will be equivalent, or better than degree 6 or 7 unweighted. For gentle curves within a band of $\frac{\rho_{n+1}}{\rho_n}$ ratios between 0.1 and 10 no problems should

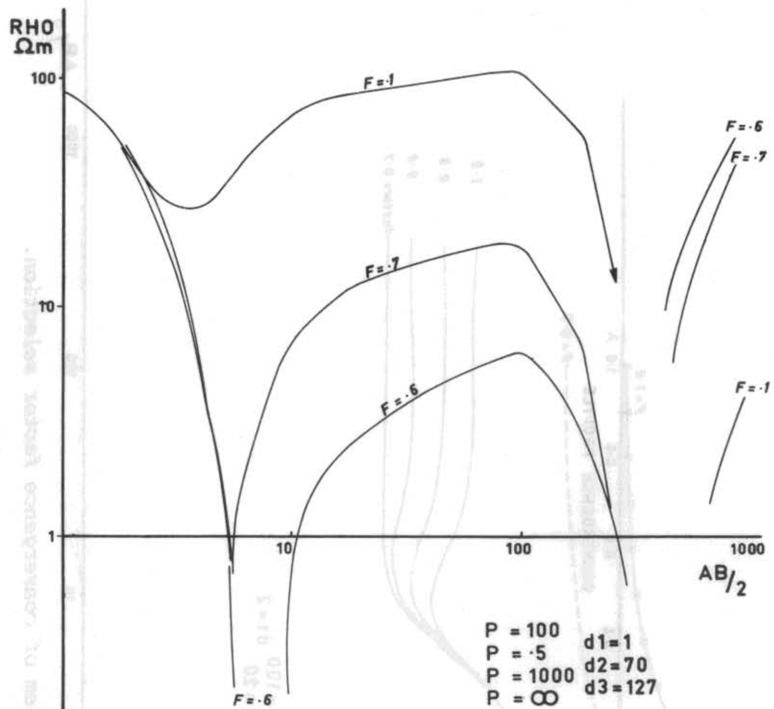

Figure 35. *Example of convergence failures.*

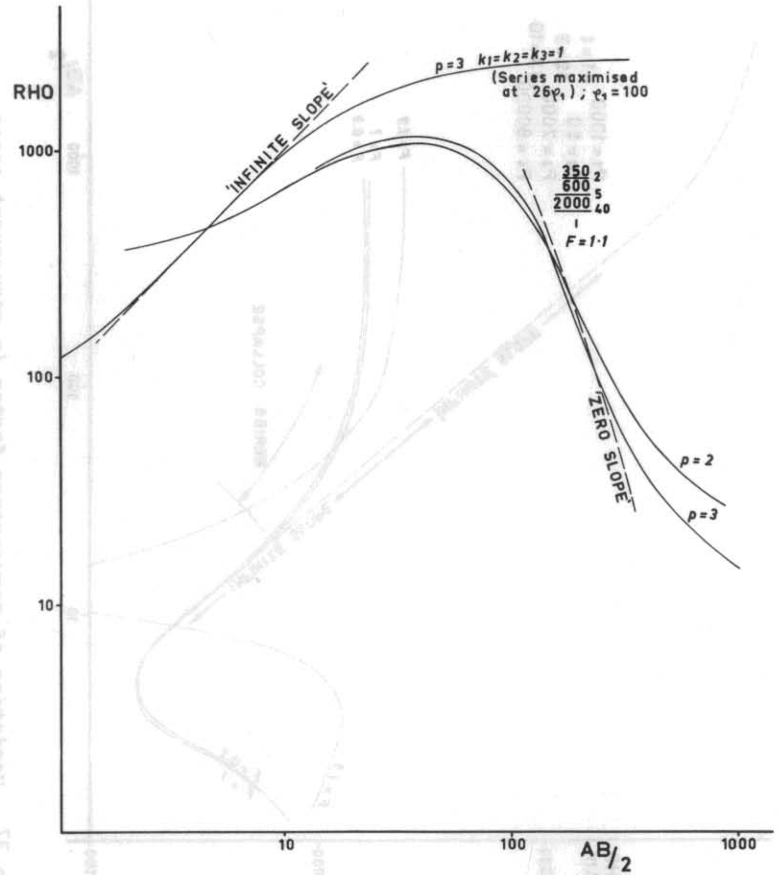

Figure 36. *Response of series to extreme values.*

5 cm

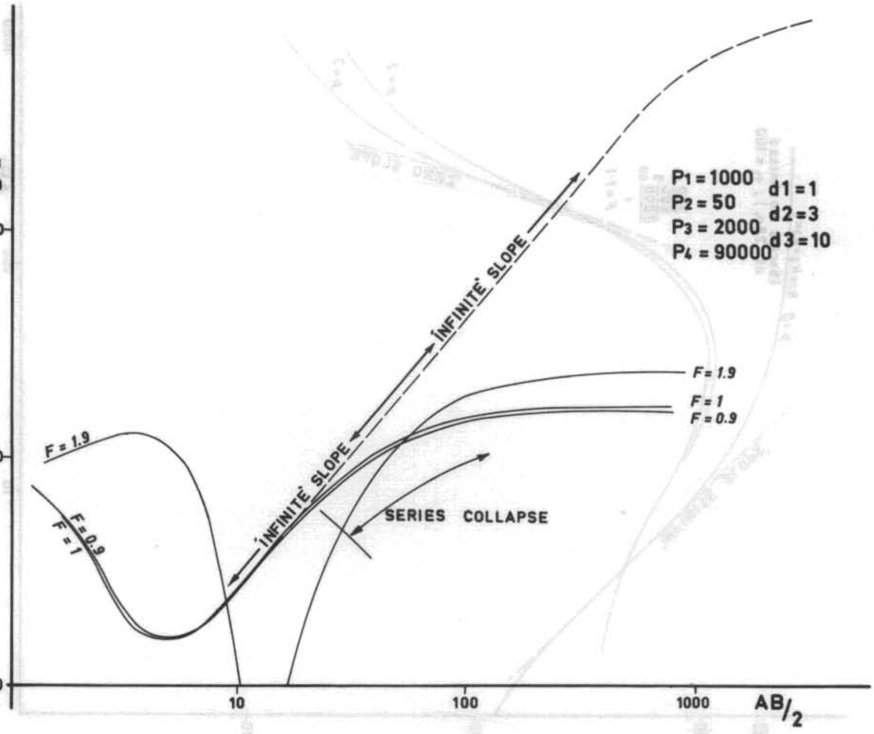

Figure 37. *Variation of convergence factor in convergent cases.*

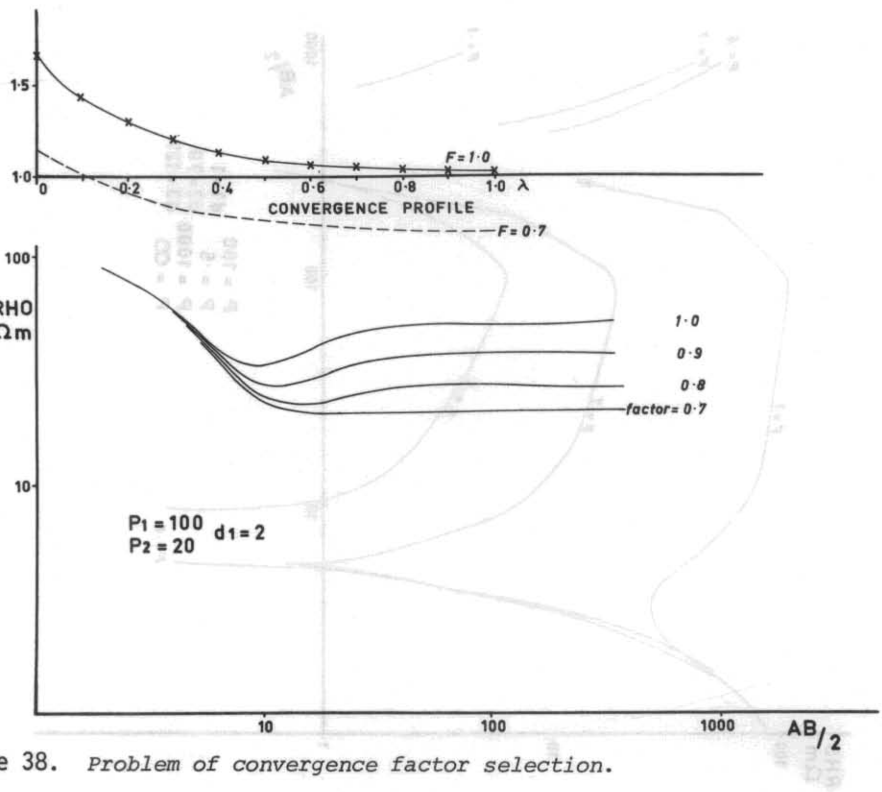

Figure 38. *Problem of convergence factor selection.*

5 cm

arise unless as mentioned above both k_1, k_2 or k_2, k_3 are large and negative. Divergent cases can be treated in the manner suggested by Van Dam by inserting a zero thickness layer of resistivity given by

$$\rho_{n+1} = \frac{1+k_n}{1-k_n} . \rho_n$$

The suitability of the convergence factor can be gauged by the convergence to the value of ρ_4 provided that such a value is within the calculation range of the series being used.

(2) The series is composed of positive and negative terms of the form $K\ (\frac{1}{\sqrt{(D/r)^2+1}})$. Now as $r \to \infty$ the D part of each term tends to 1. Thus the ultimate result is a function of the algebraic summation of the K terms. There is thus a simple limit to the total of which the series is capable at any given order of approximation (see below), *i.e.* when $k_1 = k_2 = k_3 = 1$. Thus infinite curves cannot be plotted out with this method although the infinite slope can be determined asymptotically. The high ρ_4 slope is sketched on Figures 36, 37. This slope can easily be determined by calculation of a two layer case and then extending the straight section of the line produced. Thus highly resistive cases can be allowed for. This problem only arises for the lowermost layer. A similar condition arises where $\rho_n = 0$. In this situation $k_{n-1} \to -1$. The slope condition can be estimated asymptotically although the expansion is incapable of continuous calculation along the asymptote (see fig. 36).

(3) Examination of various situations suggests that it is possible to use the second order approximation generally and obtain curves of adequate reliability. Higher order approximations appear to affect only the peak or low values and then by less than 1-3% unless there are extreme values (fig. 36). In view of calculation speed it is generally wasteful to use the $p = 3$ form of the series unless a check is being made, or the model presents special problems. In extreme cases (*e.g.* $k_1 = k_1 = k_3 = 1$) the series for $p = 2$ maximises at $16\rho_1$ whereas for $p = 3$ the maximum possible calculated value is $25\rho_1$.

Only one other problem has arisen in calculations but this is due to the inability of the Wang 700B to handle numbers in excess of 10^{99}. This situation may arise for $2d_3$-$2d_1$, or $2d_3$-$2d_2$ greater than 227 in the convergence test. However such a failure normally only affects two, or at most three, results of the convergence test and does not invalidate factor selection. By altering d_3 such that the two terms do not exceed 227 the complete convergence will be printed. Unless d_3 is of the order of d_2 or d_1 the test is unaffected, and any such alteration should be to a figure near the limit.

The calculation could be simply varied to provide for Wenner or other configurations by re-arrangement of the sub-routine which calculates the K, D term from the particular K and D values proposed at each stage.

In conclusion, the method has been found satisfactory in real situations and is capable of wide range calculation.

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